Topic 8 –
Linear Transformations and Eigenvalues

Def: Given two sets A and B a **function** f from A to B is a rule that assigns to each x in A a unique element $f(x)$ in B.

We write $f : A \to B$.

name of function

input to function

output of function lies in B

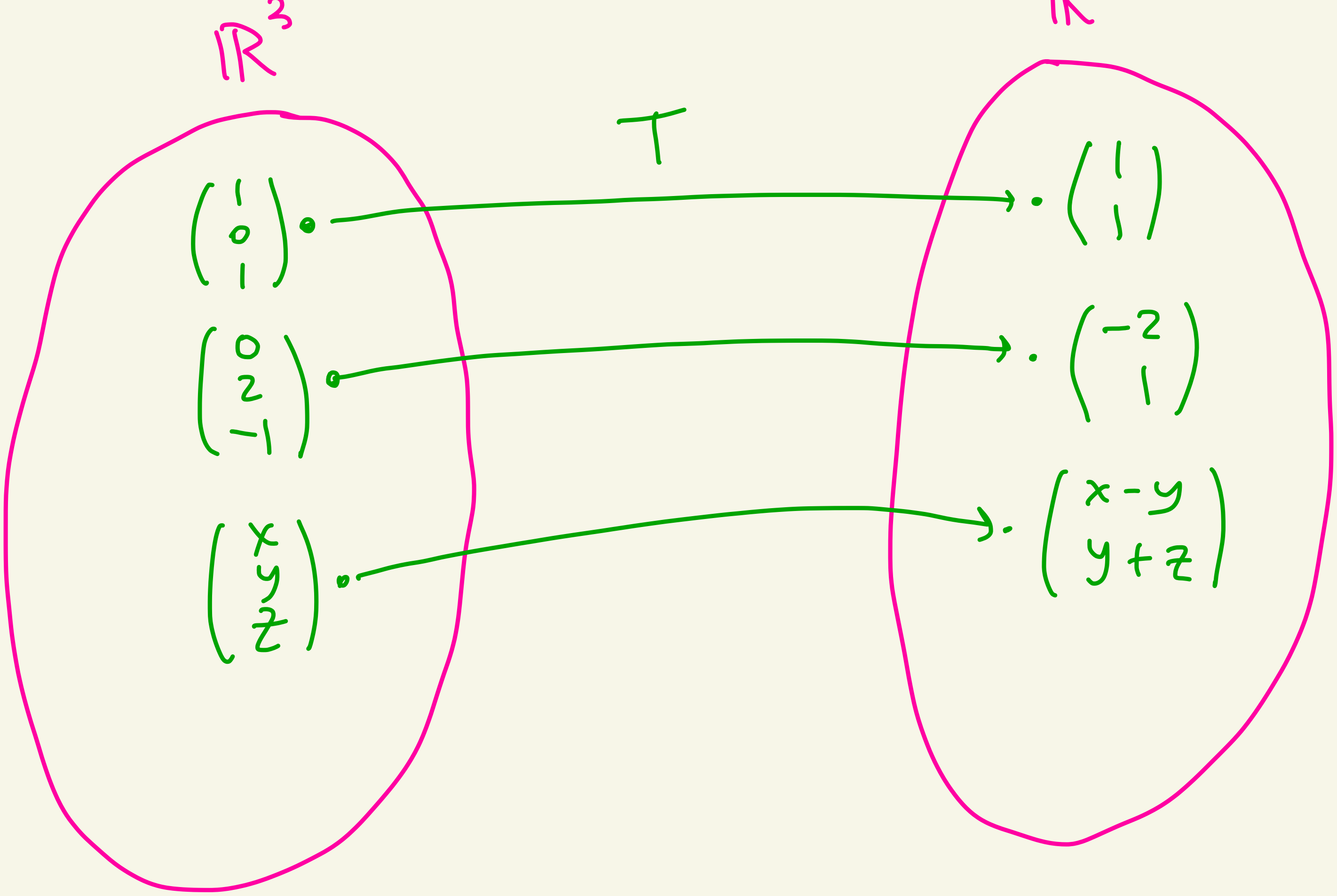
Ex: Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \end{pmatrix}$

Some calculations using T are

$$T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Here is a picture of T.



Note that

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x-y\\y+z\end{pmatrix} = \begin{pmatrix}1 & -1 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

So another formula for T is

$$T(\vec{v}) = A\vec{v}$$

where $A = \begin{pmatrix}-1 & 1 & 0\\0 & 1 & 1\end{pmatrix}$ and $\vec{v} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function of the form $T(\vec{v}) = \underbrace{A\vec{v}}$ where A is an $m \times n$ matrix.

matrix multiplication where you think of $\vec{v}$ as an $n \times 1$ matrix

Ex: Our first example above $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \end{pmatrix}$ since is a linear transformation since

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

as we derived above.

Note: When we defined $T(\vec{v}) = A\vec{v}$ we mean $\vec{v}$ is in standard coordinates. Later we will talk about how to change A to other coordinate systems.

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then by def, T is a linear transformation.

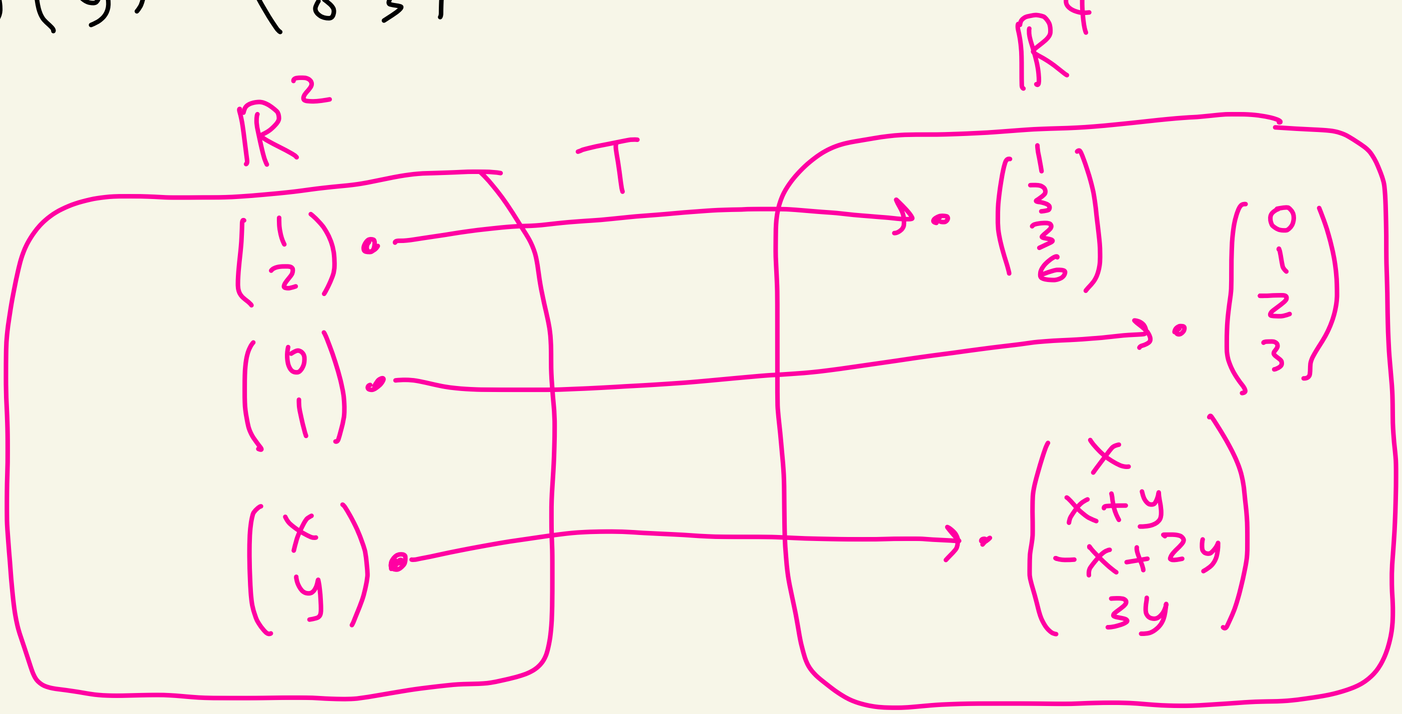
Some example calculations are:

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 2 \\ 1\cdot 1 + 1\cdot 2 \\ -1\cdot 1 + 2\cdot 2 \\ 0\cdot 1 + 3\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 6 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 0 + 0\cdot 1 \\ 1\cdot 0 + 1\cdot 1 \\ -1\cdot 0 + 2\cdot 1 \\ 0\cdot 0 + 3\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Note that in general

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 0y \\ x + y \\ -x + 2y \\ 0x + 3y \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ -x+2y \\ 3y \end{pmatrix}$$



Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$① \ T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$$

and $② \ T(\alpha \vec{v}) = \alpha T(\vec{v})$

for every scalar α and vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^n$

Conversely, a function $T: \mathbb{R}^n \to \mathbb{R}^m$ that satisfies ① and ② is a linear transformation.

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proof: Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ where $T(\vec{v}) = A\vec{v}$ then

If $\vec{v}, \vec{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then

$$① \ T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w})$$

$$② \ T(\alpha \vec{v}) = A(\alpha \vec{v}) = \alpha (A\vec{v}) = \alpha T(\vec{v})$$

Conversely suppose T satisfies ① & ②.

Let $\vec{e}_i, \vec{f}_i$ be the standard bases for $\mathbb{R}^n$ & $\mathbb{R}^m$.

Suppose $T(\vec{e}_j) = a_{1j}\vec{f}_1 + a_{2j}\vec{f}_2 + \cdots + a_{mj}\vec{f}_j$.

Then,

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$\blacksquare$

Ex: Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and you know that $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$

Find A where $T(\vec{x}) = A\vec{x}$.

Note that

$$T\begin{pmatrix}x\\y\end{pmatrix} = T\left(\begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix}\right)$$

$$= T\begin{pmatrix}x\\0\end{pmatrix} + T\begin{pmatrix}0\\y\end{pmatrix}$$

property ① above

$$= T\left(x\begin{pmatrix}1\\0\end{pmatrix}\right) + T\left(y\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= xT\begin{pmatrix}1\\0\end{pmatrix} + yT\begin{pmatrix}0\\1\end{pmatrix}$$

property ② above

$$= x\begin{pmatrix}2\\1\end{pmatrix} + y\begin{pmatrix}-1\\3\end{pmatrix}$$

$$= \begin{pmatrix}2x - y\\x + 3y\end{pmatrix} = \begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

So, $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}$

Sometimes we want to find a basis that simplifies a linear transformation. One way to do this is to try to find the "eigenvalues" and "eigenvectors" of a linear transformation.

These are when $T(\vec{v}) = \lambda \vec{v}$

where λ is a number.

Here we will only look at $T(\vec{v}) = A\vec{v}$ where A is a square $n \times n$ matrix.

<u>Def:</u> Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation be defined by $T(\vec{v}) = A\vec{v}$ where A is an $n \times n$ matrix.

Suppose that $\vec{v}$ is a vector in $\mathbb{R}^n$ with

① $\vec{v} \neq \vec{0}$

and ② $T(\vec{v}) = \lambda \vec{v}$

for some scalar λ in $\mathbb{R}$.

Then, λ is called an <u>eigenvalue</u> of T and $\vec{v}$ is called an <u>eigenvector</u> of T corresponding to λ.

Given an eigenvalue λ of T, the <u>eigenspace</u> of T corresponding to λ is

$$E_\lambda(T) = \{ \vec{v} \mid T(\vec{v}) = \lambda \vec{v} \}$$
$$= \{ \vec{v} \mid A\vec{v} = \lambda \vec{v} \}$$

<u>Note:</u> $E_\lambda(T)$ consists of all eigenvectors corresponding to λ and also includes $\vec{0}$ to make it a subspace.

<u>Note:</u> If $A\vec{v} = \lambda\vec{v}$ where $\vec{v} \neq \vec{0}$ we also call λ an eigenvalue of A and $\vec{v}$ an eigenvector of A and write $E_\lambda(A) = \{ \vec{v} \mid A\vec{v} = \lambda\vec{v} \}$

Ex: Let $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$

be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10x - 9y \\ 4x - 2y \end{pmatrix}$$

Note that

$$T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \cdot 3 - 9 \cdot 2 \\ 4 \cdot 3 - 2 \cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 30 - 18 \\ 12 - 4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

$$= 4 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Thus,

$$T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 4 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\boxed{\begin{gathered} T(\vec{v}) = \lambda \vec{v} \\ A\vec{v} = \lambda \vec{v} \end{gathered}}$$

So, $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector of T with eigenvalue $\lambda = 4$.

$\boxed{\text{How do we find the eigenvalues of a linear transformation } T\text{? That is, of an } n \times n \text{ matrix } A\text{?}}$

Suppose λ is an eigenvalue of A and $\vec{x} \neq \vec{0}$ is an eigenvector associated with λ.

Then, $A\vec{x} = \lambda\vec{x}$.

So, $A\vec{x} - \lambda\vec{x} = \vec{0}$.

using $I_n\vec{x} = \vec{x}$

Then, $(A - \lambda I_n)\vec{x} = \vec{0}$ where I_n is the $n \times n$ identity matrix.

So, $(A-\lambda I_n)\vec{x}=\vec{0}$ where $\vec{x}\neq\vec{0}$.

The only way this can happen is if $A-\lambda I_n$ has no inverse.

$\boxed{\text{Why?}}$ Let $B=A-\lambda I_n$.

If B^{-1} existed then since $B\vec{x}=\vec{0}$ you would get $B^{-1}B\vec{x}=B^{-1}\vec{0}$ which would give $\vec{x}=\vec{0}$. But $\vec{x}\neq\vec{0}$. So, B^{-1} does not exist

Thus, $\det(A-\lambda I_n)=0$ since $(A-\lambda I_n)^{-1}$ does not exist.

Summary: The eigenvalues of A satisfy the equation

$$\underbrace{\det(A - \lambda I_n) = 0.}_{\text{called the characteristic polynomial of } A}$$

Note: Henceforth we will just say eigenvalues/eigenvectors of A instead of T since its the same.

$\boxed{\text{Ex:}}$ (HW problem)

Let $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

Let's find the eigenvalues of A

Eigenvalue time!

characteristic poly.

$$\det(A - \lambda I_2) =$$

$$= \det\left(\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \det\left(\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{pmatrix}$$

$$= (10-\lambda)(-2-\lambda) - (-9)(4)$$

$$= -20 - 10\lambda + 2\lambda + \lambda^2 + 36$$

$$= \lambda^2 - 8\lambda + 16$$

$$= (\lambda - 4)(\lambda - 4)$$

$$= (\lambda - 4)^2$$

The eigenvalues of A are when $(\lambda-4)^2 = 0$.
Thus, the only eigenvalue of A is $\lambda = 4$.

Facts/Defs

Let A be an $n \times n$ matrix.
Let λ be eigenvalue of A.

① The eigenspace $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

② The dimension of $E_\lambda(A)$ is called the <u>geometric multiplicity of λ</u>.

③ The <u>algebraic multiplicity of λ</u> is the multiplicity of λ as a root of the characteristic polynomial of A.

④ $\begin{pmatrix} \text{geometric multiplicity} \\ \text{of } \lambda \end{pmatrix} \leq \begin{pmatrix} \text{algebraic} \\ \text{multiplicity} \\ \text{of } \lambda \end{pmatrix}$

Ex: Let

$$A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$$

be as in the previous example.
We had that the characteristic poly of A was

$$\det(A-\lambda I) = (\lambda-4)^2$$

Thus, $\lambda=4$ is an eigenvalue with algebraic multiplictity of 2.

Let's now find the eigenvectors corresponding to $\lambda=4$.

Let's get a basis for

$$E_4(A) = \{ \vec{x} \mid A\vec{x} = 4\vec{x} \}$$

Need to solve $A\vec{x} = 4\vec{x}$.

Let's solve!

$$\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix} \leftarrow A\vec{x} = 4\vec{x}$$

$$\begin{pmatrix} 10a - 9b \\ 4a - 2b \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \end{pmatrix}$$

$$\begin{pmatrix} 6a - 9b \\ 4a - 6b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives:

$$\boxed{\begin{aligned} 6a - 9b &= 0 \\ 4a - 6b &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right) \xrightarrow{\frac{1}{6}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So we get:

$$\boxed{\begin{aligned} a - \tfrac{3}{2}b &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a

free: b

Solutions:

$$\boxed{\begin{aligned} b &= t \\ a &= \tfrac{3}{2}b = \tfrac{3}{2}t \end{aligned}}$$

Thus if $\vec{x}$ solves $A\vec{x} = 4\vec{x}$

then $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3/2\, t \\ t \end{pmatrix} = t\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

Thus, a basis for $E_4(A)$ is $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$. Thus, $\lambda = 4$ has geometric multiplicity $\dim(E_4(A)) = 1$.

Summary table for A

eigenvalue λ	alg. mult. of λ	basis for $E_\lambda(A)$	Geometric mult. of λ
$\lambda = 4$	2	$\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$	1

What does it mean that $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$ is a basis for the eigenspace for $\lambda = 4$?

It means you can get all the eigenvectors for $\lambda = 4$ by scaling $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$ by a non-zero number.

t	eigenvector for $\lambda=4$ $t\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$
1	$\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$
-2	$\begin{pmatrix} -3 \\ -2 \end{pmatrix}$
6	$\begin{pmatrix} 9 \\ 6 \end{pmatrix}$
$\vdots$	$\vdots$

You can calculate $t = 0$ which gives

$0 \cdot \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is in $E_4(A)$

but it isn't an eigenvector

Ex: Let $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Find the eigenvalues, bases for the eigenspaces, and algebraic/geometric multiplicities of the eigenvalues.

Eigenvalue time!

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-1-\lambda) - (0)(8)$$

$$= (3-\lambda)(-1-\lambda)$$

$$= [-(\lambda-3)][-(\lambda+1)]$$

$$= (\lambda-3)^1(\lambda+1)^1$$

And $(\lambda-3)(\lambda+1)=0$

when $\lambda = 3, -1$.

So the eigenvalues are $\lambda = 3, -1$.

The algebraic multiplicity of both eigenvalues is 1.

Let's find a basis for the

eigenspace $E_3(A)$ for $\lambda = 3$.

Need to solve $A\vec{x} = 3\vec{x}$.

Need to solve

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \quad \leftarrow A\vec{x} = 3\vec{x}$$

$$\begin{pmatrix} 3a \\ 8a - b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 8a - 4b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Need to solve

$$\boxed{\begin{aligned} 8a - 4b &= 0 \\ 0 &= 0 \end{aligned}} \xrightarrow{\frac{1}{8}R_1 \to R_1} \boxed{\begin{aligned} a - \tfrac{1}{2}b &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a
free: b

Solution:

$$\boxed{\begin{aligned} b &= t \\ a &= \tfrac{1}{2} b = \tfrac{1}{2} t \end{aligned}}$$

So, if $\vec{x}$ solves $A\vec{x} = 3\vec{x}$ then

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1/2\, t \\ t \end{pmatrix} = t \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

Thus a basis for $E_3(A)$ is $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ and so $\lambda = 3$ has geometric multiplicity

$$\dim(E_3(A)) = \underbrace{1}_{\text{\# vectors in basis}}$$

Let's now find a basis for the eigenspace $E_{-1}(A)$ for $\lambda = -1$.

We need to solve $A\vec{x} = -\vec{x}$.

Need to solve

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \quad \leftarrow A\vec{x} = -\vec{x}$$

$$\begin{pmatrix} 3a \\ 8a - b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

$$\begin{pmatrix} 4a \\ 8a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} 4a &= 0 \\ 8a &= 0 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 4 & 0 & 0 \\ 8 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 8 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-8R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So we get

$$\boxed{\begin{aligned} a &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a

free: b

Solution:

$$\boxed{\begin{aligned} b &= t \\ a &= 0 \end{aligned}}$$

So, if $\vec{x}$ solves $A\vec{x} = -\vec{x}$ then

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So a basis for $E_{-1}(A)$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and so $\lambda = -1$ has geometric mult. $\dim(E_{-1}(A)) = 1$

vectors in basis

Summary for $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

eigenvalue λ	algebraic mult.	basis for $E_\lambda(A)$	geometric mult.
$\lambda = 3$	1	$\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$	1
$\lambda = -1$	1	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1

Ex: Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Let's find the eigenvalues of A. We need to solve

$$\det(A - \lambda I_3) = 0$$

because A is 3x3

We have

$$\det(A-\lambda I_3)$$

$$= \det\left(\underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_{A} - \lambda \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3}\right)$$

$$= \det\left(\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

expand on column 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -0 + (2-\lambda)\underbrace{\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}}_{\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}} - 0$$

$$= (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(-\lambda)(3-\lambda) - (1)(-2)\right]$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2)$$

$$= (2-\lambda)(\lambda-1)(\lambda-2)$$

$$= -(\lambda-2)(\lambda-1)(\lambda-2)$$

$$= -(\lambda-2)^2(\lambda-1)$$

$\lambda = 2$ has alg. mult. 2

The eigenvalues are $\lambda = 2, 1$. $\lambda = 1$ has alg. mult. 1.

Let's find the eigenvectors of A.

Let's start with $\lambda = 1$.

Let's find a basis for

$$E_1(A) = \{ \vec{x} \mid A\vec{x} = 1 \cdot \vec{x} \}$$

The equation $A\vec{x} = 1 \cdot \vec{x}$ becomes

$$\underbrace{\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{A\vec{x}} = \underbrace{1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{1 \cdot \vec{x}}$$

This becomes

$$\begin{pmatrix} 0a + 0b - 2c \\ a + 2b + c \\ a + 0b + 3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives
$$\begin{pmatrix} -2c \\ a+2b+c \\ a \quad +3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives
$$\begin{pmatrix} -a \quad -2c \\ a+b+c \\ a \quad +2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,
$$\boxed{\begin{aligned} -a \quad -2c &= 0 \\ a+b+c &= 0 \\ a \quad +2c &= 0 \end{aligned}}$$

Solving we get
$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We get

$$\boxed{\begin{aligned} a \quad\quad + 2c &= 0 \\ b - c &= 0 \\ 0 &= 0 \end{aligned}} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

leading: a, b
free: c

Solving:

$$\boxed{\begin{aligned} &c = t \\ (2)\ &b = c = t \\ (1)\ &a = -2c = -2t \end{aligned}}$$

Thus, if $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in $E_1(A)$

then $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

So, $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is a basis for $E_1(A)$

Thus, $\dim(E_1(A)) = 1$

Let's now find a basis for

$$E_2(A) = \{ \vec{x} \mid A\vec{x} = 2\vec{x} \}$$

Want to solve $A\vec{x} = 2\vec{x}$.

So need to solve

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \leftarrow A\vec{x} = 2\vec{x}$$

$$\begin{pmatrix} -2c \\ a + 2b + c \\ a + 3c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

$$\begin{pmatrix} -2a - 2c \\ a + c \\ a + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} -2a - 2c &= 0 \\ a + c &= 0 \\ a + c &= 0 \end{aligned}}$$

Let's solve:

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives:

$$\boxed{\begin{aligned} a \qquad + c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a

free: c, b

Solution:

$$\boxed{\begin{aligned} b &= t \\ c &= u \\ a &= -c = -u \end{aligned}}$$

Thus, if $\vec{x}$ solves $A\vec{x} = 2\vec{x}$ then

$$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -u \\ t \\ u \end{pmatrix}$$

$$= \begin{pmatrix} -u \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$$

$$= u \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So all solutions of $A\vec{x} = 2\vec{x}$ are linear combinations of $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Thus, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ span the eigenspace $E_2(A)$.

You can verify that $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

Thus, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis for $E_2(A)$. So, $\lambda = 2$ has geometric multiplicity $\dim(E_2(A)) = 2$.

Summary table for A:

eigenvalue λ	alg. mult. of λ	basis for $E_\lambda(A)$	geometric mult.
$\lambda = 1$	1	$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$	1
$\lambda = 2$	2	$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	2